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Note

Largest subsemigroups of the full transformation monoid

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Abstract

In this paper we are concerned with the following question: for a semigroup S , what is the largest size of a subsemigroup $T \leq S$ where T has a given property? The semigroups S that we consider are the full transformation semigroups; all mappings from a finite set to itself under composition of mappings. The subsemigroups T that we consider are of one of the following types: left zero, right zero, completely simple, or inverse. Furthermore, we find the largest size of such subsemigroups U where the least rank of an element in U is specified. Numerous examples are given.

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1. Introduction

In this paper we are concerned with questions of the type: for a semigroup S , what is the largest size of a subsemigroup $T \leq S$ where T has a given property? The semigroups S that we consider are the full transformation semigroups \mathcal{T}_n ; all mappings from the set $\{1, 2, \dots, n\}$ to itself under composition of mappings. The different types of subsemigroup that we consider are among the most fundamental kinds of semigroup. Starting with *left and right zero semigroups*, i.e. semigroups U satisfying $xy = x$ and $xy = y$, respectively, for all $x, y \in U$. Moving on to the more general class of *completely simple semigroups*; those semigroups where every element lies in a subgroup and that have no proper ideals. Of course, left and right zero semigroups are examples of completely simple semigroups. However, it turns out that the largest completely simple subsemigroups of \mathcal{T}_n are not left or right zero semigroups when $n \geq 3$. As such the three types of semigroups are considered separately. Finally, we consider semigroups U such that for all $x \in U$ there exists a unique $y \in U$ such that $xyx = x$ and $yxy = y$; called *inverse semigroups*. Since every finite semigroup can be embedded in \mathcal{T}_n for some n , there are nonempty subsemigroups of \mathcal{T}_n of each of these types.

It is not difficult to see that the largest proper subsemigroup U of \mathcal{T}_n , without any further property, is the union of all the noninvertible mappings in \mathcal{T}_n and the alternating subgroup of \mathcal{S}_n ; see, for example, [7] or [10]. This semigroup has size $n^n - n!/2$. The semigroup U happens to be *regular*; that is, for all $x \in U$ there exists $y \in U$ such that $xyx = x$. Thus the largest size of a proper regular subsemigroup of \mathcal{T}_n is also $n^n - n!/2$. In [9] and [10], similar results are given for both subsemigroups and regular subsemigroups of the semigroup of all mappings in \mathcal{T}_n with image size at most r . Several authors have considered *maximal*, with respect to containment, inverse subsemigroups of \mathcal{T}_n ; see [5] and [6].

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However, the largest size of an inverse subsemigroup seems not to have been considered. A classification of maximal inverse subsemigroups of \mathcal{T}_n remains to be found.

Questions of the type considered in this paper, have been considered for subgroups, of various types, of the symmetric group; see, for example, [1,4], or [8]. Incidentally, the size of the largest abelian subgroups of \mathcal{S}_n coincides with the size of the largest right zero subsemigroups of \mathcal{T}_n ; see [1] and Theorem 3.4.

In the remainder of this section, we give the definitions and notation required in the rest of the paper. Throughout we assume that 0 is an element of \mathbb{N} . We write mappings with their argument on the left, and hence compose from left to right. The *image* of an element $\alpha \in \mathcal{T}_n$ is, as usual, denoted by $\text{im}(\alpha) = \{j : \text{there exists } i \text{ such that } i\alpha = j\}$ and the *kernel* of α is denoted by $\ker(\alpha) = \{(i, j) : i\alpha = j\alpha\}$. The *rank* of an element $\alpha \in \mathcal{T}_n$ is the size of its image; denoted by $\text{rank}(\alpha)$. For a subsemigroup U of \mathcal{T}_n denote by $\text{Ims}(U)$ and $\text{Kers}(U)$ the set of all those images and kernels, respectively, that elements of U admit. If S is a semigroup, then we denote S with an identity adjoined by S^1 , keeping the convention that if S has an identity, then $S^1 = S$. For a subset X of $\{1, 2, \dots, n\}$, the *cone* of X in U , denoted $\text{Cone}_U(X)$, is the set $\{X\alpha : \alpha \in U^1\}$. A set is a *transversal* of a partition if it intersects each class in exactly one element. A *partial transversal* is just a subset of a transversal.

Two elements x, y of a semigroup S are said to be \mathcal{L} -related if they generate the same left ideal, that is, $S^1x = S^1y$; we write $x\mathcal{L}y$ instead of $(x, y) \in \mathcal{L}$. The relation \mathcal{R} is defined analogously, replacing left ideals with right ideals. Two elements are \mathcal{H} -related if they are both \mathcal{L} - and \mathcal{R} -related and the relation \mathcal{D} is defined to be the composition of \mathcal{L} and \mathcal{R} as binary relations. Recall that two elements $\alpha, \beta \in \mathcal{T}_n$ are \mathcal{L} -related if and only if $\text{im}(\alpha) = \text{im}(\beta)$, \mathcal{R} -related if and only if $\ker(\alpha) = \ker(\beta)$, and \mathcal{D} -related if and only if $\text{rank}(\alpha) = \text{rank}(\beta)$. Furthermore, if U is a regular subsemigroup of \mathcal{T}_n , then Green's \mathcal{L} and \mathcal{R} relations on U are just the restrictions of these relations on \mathcal{T}_n to $U \times U$. If $\alpha \in \mathcal{T}_n$, then we denote the \mathcal{L} -class containing α by L_α . Likewise, the \mathcal{R} -class, \mathcal{H} -class and \mathcal{D} -class containing α are denoted by R_α , H_α and D_α , respectively. At times, it may be necessary to distinguish between Green's relations on a subsemigroup U and those of the parent semigroup \mathcal{T}_n . To this end, we write \mathcal{L}^U , \mathcal{R}^U , \mathcal{H}^U and \mathcal{D}^U to mean Green's \mathcal{L} -, \mathcal{R} -, \mathcal{H} - and \mathcal{D} -relations on U , respectively. Similarly, we write L_α^U to denote the \mathcal{L} -class of α in U . The classes R_α^U , H_α^U and D_α^U are defined analogously. We denote by $D(r)$ the \mathcal{D} -class of all the elements in \mathcal{T}_n of rank r . For more information on Green's relations see [3].

2. Properties of semigroups via generating sets

In this section we give a series of results that allow us to determine whether a set of mappings in \mathcal{T}_n generates a semigroup of a given type. These types are: left zero semigroups, right zero semigroups, inverse semigroups, groups, completely simple semigroups, completely regular semigroups, and Clifford semigroups (although later on we only require Propositions 2.1, 2.3, and 2.6). At first glance it might not be obvious why the conditions of the results in this section are an improvement over the original definitions. The main point is that it can be easily determined whether a subsemigroup $U = \langle \Omega \rangle$, with $\Omega \subseteq \mathcal{T}_n$, satisfies these conditions by considering Ω only. For example, it is possible to compute $\text{Ims}(U)$ and $\text{Kers}(U)$, in Proposition 2.6, using a simple orbit algorithm, without computing all the elements of U .

The following two propositions are routine and the proofs are omitted.

Proposition 2.1. *Let $\Omega \subseteq \mathcal{T}_n$ and $U = \langle \Omega \rangle$. Then U is a left zero semigroup if and only if for all $\alpha, \beta \in \Omega$, $\text{im}(\alpha) = \text{im}(\beta)$ and $\alpha^2 = \alpha$.*

On the other hand, U is a right zero semigroup if and only if for all $\alpha, \beta \in \Omega$, $\ker(\alpha) = \ker(\beta)$ and $\alpha^2 = \alpha$.

Note that if U is a left or right zero semigroup, then $U = \langle \Omega \rangle$ if and only if $U = \Omega$.

Proposition 2.2. *Let $\Omega \subseteq \mathcal{T}_n$ and $U = \langle \Omega \rangle$. Then U is a group if and only if for all $\alpha, \beta \in \Omega$*

- (i) $\text{im}(\alpha) = \text{im}(\beta)$ and $\alpha|_{\text{im}(\alpha)}$ is a permutation;
- (ii) $\ker(\alpha) = \ker(\beta)$.

It is straightforward to check that a subsemigroup U of \mathcal{T}_n is completely simple if and only if all the elements of U have the same rank.

Proposition 2.3. Let $\Omega \subseteq \mathcal{T}_n$ and $U = \langle \Omega \rangle$. Then U is completely simple if and only if for all $\alpha, \beta \in \Omega$, $\text{im}(\alpha)$ is a transversal of $\ker(\beta)$.

Proof. For the direct implication, U is completely simple and so $\text{rank}(\alpha) = \text{rank}(\beta)$ for all $\alpha, \beta \in U$. Thus for any $\alpha, \beta \in \Omega$, $\text{im}(\alpha)$ must be a transversal of $\ker(\beta)$, otherwise $\text{rank}(\alpha\beta) < \text{rank}(\alpha)$.

For the converse, by assumption, $\text{rank}(\alpha) = \text{rank}(\beta)$ for all $\alpha, \beta \in \Omega$. Let $\gamma \in U$ be arbitrary. Then we can write $\gamma = \omega_1 \omega_2 \cdots \omega_k$ for $\omega_1, \omega_2, \dots, \omega_k \in \Omega$. Since $\text{im}(\omega_i)$ is a transversal of $\ker(\omega_{i+1})$ for all i , it follows that $\text{im}(\gamma) = \text{im}(\omega_k)$. Thus every element of U has the same rank, and so U is completely simple. \square

A semigroup is *completely regular* if every element lies in a subgroup. A subsemigroup U of \mathcal{T}_n is completely regular if and only if $\text{rank}(\alpha^2) = \text{rank}(\alpha)$ for all $\alpha \in U$.

Proposition 2.4. Let $\Omega \subseteq \mathcal{T}_n$ and $U = \langle \Omega \rangle$. Then U is completely regular if and only if for all $\alpha \in \Omega$, $\text{Cone}_U(\text{im}(\alpha))$ consists of partial transversals of $\ker(\alpha)$.

Proof. For the direct implication, let $\alpha \in \Omega$ and $\beta \in U$. Since $\text{rank}((\alpha\beta)^2) = \text{rank}(\alpha\beta)$ it follows that $\text{im}(\alpha\beta)$ is a transversal of $\ker(\alpha\beta)$. This implies that $\text{im}(\alpha\beta)$ is a partial transversal of $\ker(\alpha)$.

For the converse implication, let $\alpha \in U$ be arbitrary. Then $\alpha = \omega_1 \omega_2 \cdots \omega_m$ for some $\omega_1, \omega_2, \dots, \omega_m \in \Omega$. The image of $\omega_1 \cdots \omega_m$ lies in the cone of $\text{im}(\omega_1)$ and so it is a partial transversal of $\ker(\omega_1)$. Therefore $\text{rank}(\omega_1 \cdots \omega_m \omega_1) = \text{rank}(\omega_1 \cdots \omega_m)$. Let $k \in \{1, \dots, m-1\}$. Then $\text{im}(\omega_1 \cdots \omega_m \omega_1 \cdots \omega_k) \subseteq \text{im}(\omega_{k+1} \cdots \omega_m \omega_1 \cdots \omega_k)$ which is an element of $\text{Cone}_U(\text{im}(\omega_{k+1}))$. It follows that $\text{im}(\omega_1 \cdots \omega_m \omega_1 \cdots \omega_k)$ is a partial transversal of $\ker(\omega_{k+1})$, and so $\text{rank}(\omega_1 \cdots \omega_m \omega_1 \cdots \omega_{k+1}) = \text{rank}(\omega_1 \cdots \omega_m \omega_1 \cdots \omega_k)$. At the end of this process we may conclude that $\text{rank}(\alpha^2) = \text{rank}(\alpha)$. \square

Next, we consider inverse subsemigroups of \mathcal{T}_n . The result in this case is somewhat weaker than those above, because we must assume that our subsemigroup is known to be regular. We require the following straightforward lemma before moving on to the main result about inverse semigroups.

Lemma 2.5. Let U be a regular subsemigroup of \mathcal{T}_n , let $I \in \text{Ims}(U)$ and $K \in \text{Kers}(U)$ such that I is a transversal of K . Then there exists an idempotent $\varepsilon \in U$ such that $\text{im}(\varepsilon) = I$ and $\ker(\varepsilon) = K$.

Proof. Let $\alpha, \beta \in U$ such that $\text{im}(\alpha) = I$ and $\ker(\beta) = K$. Then $\ker(\alpha\beta) = \ker(\alpha)$ and $\text{im}(\alpha\beta) = \text{im}(\beta)$. Since U is regular we deduce that $\alpha\beta\mathcal{R}^U\alpha$ and $\alpha\beta\mathcal{L}^U\beta$. Thus, by [3, Proposition 2.3.7], $L_\alpha^U \cap R_\beta^U$ contains an idempotent with the required properties. \square

A semigroup is inverse if and only if every \mathcal{R} -class and \mathcal{L} -class contains exactly one idempotent, see [3, Theorem 5.1.1].

Proposition 2.6. Let U be a regular semigroup. Then U is inverse if and only if $|\text{Ims}(U)| = |\text{Kers}(U)|$ and for each $I \in \text{Ims}(U)$ there exists a unique $K \in \text{Kers}(U)$ such that I is a transversal of K .

Proof. For the direct implication, the fact that the \mathcal{D} -classes of an inverse semigroup are square implies that $|\text{Ims}(U)| = |\text{Kers}(U)|$. To prove that the second condition holds let $I \in \text{Ims}(U)$. Since U is regular I corresponds to an \mathcal{L} -class L_I of U . Now, L_I contains an idempotent ε and I is a transversal of $\ker(\varepsilon) \in \text{Kers}(U)$. If there exists $K \in \text{Kers}(U)$ such that I is a transversal of K , then, by Lemma 2.5, there exists an idempotent $\varepsilon' \in L_I$ with $\ker(\varepsilon') = K$. But ε is the unique idempotent in L_I , and so $\varepsilon' = \varepsilon$. Thus $K = \ker(\varepsilon)$.

The converse implication follows quickly by Lemma 2.5, since there is exactly one idempotent in every \mathcal{L} -class and \mathcal{R} -class. \square

A semigroup U is *Clifford* if and only if it is regular and its idempotents are central (i.e. for each idempotent e , $es = se$ for all $s \in U$), see [3, Theorem 4.2.1]. If U is a semigroup and $s \in U$ is any element that lies in a subgroup of U , then we denote the identity element of this subgroup by e_s . The next result was originally proved by Evseev [2] but the presented proof is ours.

Proposition 2.7. *Let $\Omega \subseteq \mathcal{T}_n$ and $U = \langle \Omega \rangle$. Then U is a Clifford semigroup if and only if for all $\alpha, \beta \in \Omega$*

- (i) $\alpha|_{\text{im}(\alpha)}$ is a permutation of $\text{im}(\alpha)$;
- (ii) $\alpha e_\beta = e_\beta \alpha$.

Proof. The forward implication follows by Proposition 2.2 and the discussion before the proposition.

For the converse implication, by condition (i), each of the elements $\omega \in \Omega$ lies in a subgroup of U , and so the elements e_ω exist. Condition (ii) implies that every element in U , being a product of elements of Ω , commutes with all the elements e_ω , $\omega \in \Omega$. Any element $\omega_1 \omega_2 \cdots \omega_k$ (where $\omega_1, \omega_2, \dots, \omega_k \in \Omega$) in U lies in $H_{e_{\omega_1} e_{\omega_2} \cdots e_{\omega_k}}$. This \mathcal{H} -class contains the idempotent $e_{\omega_1} e_{\omega_2} \cdots e_{\omega_k}$ and so $H_{e_{\omega_1} e_{\omega_2} \cdots e_{\omega_k}}$ is a group. It follows that U is (completely) regular. Finally, if $e \in U$ is an idempotent, then $e = e_{\omega_1} e_{\omega_2} \cdots e_{\omega_k}$, for some $\omega_1, \omega_2, \dots, \omega_k \in \Omega$, and hence the idempotents are central. \square

3. Left and right zero semigroups

In this section, we determine the largest left and right zero subsemigroups of \mathcal{T}_n .

Proposition 3.1. *Let U be a left zero subsemigroup of \mathcal{T}_n where the rank of the elements is r . Then $|U| \leq r^{n-r}$. In particular, if U consists of all idempotent mappings with a given image of size r , then $|U| = r^{n-r}$.*

Proof. By Proposition 2.1, it suffices to find the largest set of idempotents with a common image of size r . The size of this set is r^{n-r} . \square

For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the largest integer not greater than x and the smallest integer not less than x , respectively.

Proposition 3.2. *Let U be a right zero subsemigroup of \mathcal{T}_n where the rank of the elements is r . Then $|U| \leq \lceil n/r \rceil^t \lfloor n/r \rfloor^{r-t} \leq \lfloor (n/r)^r \rfloor$, where $0 \leq t \leq r-1$ and $n \equiv t \pmod r$.*

In particular, if U consists of all idempotent mappings with a given kernel with t classes of size $\lceil n/r \rceil$ and $r-t$ classes of size $\lfloor n/r \rfloor$, then $|U| = \lceil n/r \rceil^t \lfloor n/r \rfloor^{r-t}$.

Proof. Analogous to the proof of Proposition 3.1, it suffices to find the largest set of idempotents with a common kernel that has r classes. However, in this case, two kernels with the same number of classes may give rise to right zero semigroups with different sizes. Indeed, if K is a partition of $\{1, 2, \dots, n\}$ with r classes, then the total number of idempotents with kernel K equals the number of transversals of K . This number is the product of the sizes of the kernel classes of K .

In other words, we must determine $\max\{a_1 \cdots a_r : n = a_1 + \cdots + a_r\}$, over all partitions of n into r numbers $a_1, a_2, \dots, a_r \in \mathbb{N}$. It follows by the arithmetic mean–geometric mean inequality that this number is at most $\lfloor (n/r)^r \rfloor$. In the case that r divides n then the proof is complete.

Assume that r does not divide n . Let $b_1, b_2, \dots, b_r \in \mathbb{N}$ such that $b_1 + b_2 + \cdots + b_r = n$ and $b_1 \cdots b_r = \max\{a_1 \cdots a_r : n = a_1 + \cdots + a_r\}$. If $b_1 > \lceil n/r \rceil$, then there is at least one $b_i \leq \lfloor n/r \rfloor$. Without loss of generality assume that $b_2 \leq \lfloor n/r \rfloor$. It follows that $(b_1 - 1)(b_2 + 1) > b_1 b_2$ and so $(b_1 - 1)(b_2 + 1)b_3 \cdots b_r > b_1 b_2 \cdots b_r$, a contradiction. If $b_1 < \lfloor n/r \rfloor$, then a contradiction is obtained using an analogous argument. It follows that $b_1, b_2, \dots, b_r \in \{\lceil n/r \rceil, \lfloor n/r \rfloor\}$. If $n = x \lceil n/r \rceil + y \lfloor n/r \rfloor$, then $x = t$ and $y = r - t$. Thus $b_1 b_2 \cdots b_r = \lceil n/r \rceil^t \lfloor n/r \rfloor^{r-t}$. \square

Having found the largest left and right zero subsemigroups of \mathcal{T}_n with given image size, we now find the largest overall.

Theorem 3.3. *The largest size of a left zero subsemigroup of \mathcal{T}_n is $\max\{u^{n-u} : u \in \{\lfloor x \rfloor, \lceil x \rceil\}\}$ where x is the solution to the equation $x(1 + \ln x) = n$, $x \in \mathbb{R}$.*

Proof. The result is trivial when $n = 1$, so we assume that $n \geq 2$. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = x^{n-x}$. Then

differentiating, we obtain

$$f'(x) = x^{n-x-1}(n - x(1 + \ln x)).$$

If u is a solution to the equation $x(1 + \ln x) = n$, then f is increasing when $1 \leq x < u$ and decreasing when $x > u$. Therefore f has a maximum when $x(1 + \ln x) = n$. \square

Theorem 3.4. *Let U be a right zero subsemigroup of \mathcal{T}_n , $n \geq 2$, with largest size. Then*

$$|U| = \begin{cases} 3^{(n/3)} & \text{if } n \equiv 0 \pmod{3} \\ 2^2 \cdot 3^{(n-4)/3} & \text{if } n \equiv 1 \pmod{3} \\ 2 \cdot 3^{(n-2)/3} & \text{if } n \equiv 2 \pmod{3} \end{cases}$$

and there is a right zero subsemigroup of \mathcal{T}_n with this size contained in $D(\lceil n/3 \rceil)$.

Proof. This result follows by an argument similar to that used in the proof of Proposition 3.2. Indeed, it suffices to find the maximum value of $\lfloor (n/r)^r \rfloor$ over all r between 1 and n . Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f(x) = (n/x)^x$. Then f has a maximum when $x = n/e$ and the result follows. \square

Let L be a left zero subsemigroup of \mathcal{T}_n with largest possible size and R be a right zero subsemigroup of \mathcal{T}_n with largest possible size. The sizes of L and R are given in the table below for $n \leq 10$.

n	1	2	3	4	5	6	7	8	9	10
$ L $	1	1	2	4	9	27	81	256	1024	4096
$ R $	1	2	3	4	6	9	12	18	27	36

In general, largest right zero and largest left zero semigroups do not lie in the same \mathcal{D} -class of \mathcal{T}_n . For example, in \mathcal{T}_6 the only right zero semigroups with size 9 consist of mappings with rank 2. On the other hand, the only left zero semigroup with size 27 consist of mappings with rank 3.

4. Completely simple semigroups

In this section we determine the largest size that a completely simple subsemigroup of \mathcal{T}_n can have. The results of the previous section provide us with bounds on the size of such a semigroup. Indeed, if U is a left, or right, zero subsemigroup of \mathcal{T}_n , then $V = \{\alpha \in \mathcal{T}_n : \text{im}(\alpha) \in \text{Im}(U) \text{ \& \; } \ker(\alpha) \in \text{Ker}(U)\}$ is completely simple. Moreover, if U consists of elements of rank r , then $|V| = r!|U|$. Therefore, a completely simple subsemigroup T of \mathcal{T}_n with elements of rank r and largest possible size satisfies

$$|T| \geq \max\{r!r^{n-r}, r! \lceil n/r \rceil^t \lfloor n/r \rfloor^{r-t}\} (=r!r^{n-r} \text{ when } r \geq 2), \quad (1)$$

where $0 \leq t \leq r-1$ and $n \equiv t \pmod{r}$. (Obviously, if $r = 1$, then $|T| \leq n$). On the other hand, the number of \mathcal{L} -classes in T is at most $\lceil n/r \rceil^t \lfloor n/r \rfloor^{r-t}$ and the number of \mathcal{R} -classes is at most r^{n-r} . Thus

$$|T| \leq r!r^{n-r} \lceil n/r \rceil^t \lfloor n/r \rfloor^{r-t}. \quad (2)$$

The main results of this section show that in fact the lower bound (1) is also an upper bound.

Theorem 4.1. *Let U be a completely simple subsemigroup of \mathcal{T}_n where the rank of the elements is $r \geq 2$. Then $|U| \leq r!r^{n-r}$.*

In particular, if U consists of all mappings α with a given image of size r that satisfy $\text{rank}(\alpha^2) = \text{rank}(\alpha)$, then $|U| = r!r^{n-r}$.

As a consequence of Theorem 4.1 and the fact that

$$n! > (n-1)(n-1)! > (n-2)^2(n-2)! > (n-3)^3(n-3)! > \dots$$

we have the following result.

Theorem 4.2. *The largest size of a completely simple subsemigroup of \mathcal{T}_n where the rank of the elements is at most $r \geq 2$ is $r!r^{n-r}$. In particular, the largest completely simple subsemigroup of \mathcal{T}_n is \mathcal{S}_n .*

In the remainder of this section, we will be occupied with the task of proving Theorem 4.1. In order to do this, we reformulate the problem. Let \mathcal{A} be a non-empty set of r element subsets of $\{1, 2, \dots, n\}$. Define a (simple and undirected) graph $\Gamma_{\mathcal{A}}$ with vertex set $\{1, 2, \dots, n\}$ and edge (a, b) if $\{a, b\}$ is contained in some element of \mathcal{A} . An r -colouring of a graph Γ is a surjective mapping κ from the vertices of Γ onto an r element set C such that whenever (x, y) is an edge in Γ , we have $x\kappa \neq y\kappa$. Denote by $C_r(\Gamma)$ the set of all r -colourings of Γ with a fixed set of r colours. Throughout the following we will find it more convenient to write Γ_U than $\Gamma_{\text{Im}(U)}$.

Lemma 4.3. *Let U be a completely simple subsemigroup of \mathcal{T}_n contained in $D(r)$. Then $|U| \leq |\text{Im}(U)| |C_r(\Gamma_U)|$. Moreover, for each n and r there exist completely simple subsemigroups for which equality holds.*

Proof. Let \mathcal{K} denote the set of all kernels, i.e. partitions of $\{1, 2, \dots, n\}$, that are transversed by every set in $\text{Im}(U)$. Since U is completely simple, the number of \mathcal{L} -classes in U is $|\text{Im}(U)|$, the number of \mathcal{R} -classes in U is at most $|\mathcal{K}|$, and the size of any \mathcal{H} -class is at most $r!$. Thus $|U| \leq |\text{Im}(U)| r! |\mathcal{K}|$. It remains to prove that $r! |\mathcal{K}| = |C_r(\Gamma_U)|$.

The kernel of an r -colouring κ of Γ_U is a partition of $\{1, 2, \dots, n\}$; every element in a given class has the same colour. Moreover, by the definition of Γ_U , $\ker(\kappa)$ is transversed by every set in $\text{Im}(U)$. Thus $\ker(\kappa) \in \mathcal{K}$ and κ is one of $r!$ distinct colourings with the same kernel. It follows that $r! |\mathcal{K}| \geq |C_r(\Gamma_U)|$.

On the other hand, if $K \in \mathcal{K}$, then K is transversed by every set in $\text{Im}(U)$. In particular, no 2 elements in the same class of K are adjacent in Γ_U . Thus K gives rise to $r!$ colourings of Γ_U and so $r! |\mathcal{K}| \leq |C_r(\Gamma_U)|$. \square

The next result says that for every completely simple subsemigroup U of \mathcal{T}_n with at least two \mathcal{L} -classes we can find another completely simple semigroup, in the same \mathcal{D} -class, which is not smaller than U and with strictly fewer \mathcal{L} -classes.

Lemma 4.4. *Let U be a completely simple subsemigroup of \mathcal{T}_n contained in $D(r)$, $r \geq 2$, with $|\text{Im}(U)| \geq 2$. Then there exists a completely simple subsemigroup V of \mathcal{T}_n which is contained in $D(r)$ and satisfies $|\text{Im}(V)| < |\text{Im}(U)|$ and $|V| \geq |U|$.*

Proof. Let $\alpha \in U$. Then $\ker(\alpha)$ is transversed by $\text{im}(\beta)$ for all $\beta \in U$. Thus if we colour all the vertices in each kernel class of α with a different colour from the r -element set C , we obtain an r -colouring of Γ_U . Therefore, the largest complete subgraph of Γ_U has r vertices and, since $\text{Im}(U)$ is nonempty, there is such a complete subgraph. Denote the abstract complete graph on r vertices by K_r . Following from this and the fact that $|\text{Im}(U)| \geq 2$, there exist vertices x and y of Γ_U with nonzero degree such that (x, y) is not an edge in Γ_U . Suppose without loss of generality that the number of images containing x is no larger than the number of images containing y .

Now, let V be the largest completely simple subsemigroup of \mathcal{T}_n with $\text{Im}(V)$ equal to the set of all the elements in $\text{Im}(U)$ that do not contain x . By Lemma 4.3, $|V| = |\text{Im}(V)| |C_r(\Gamma_V)|$. Since x is contained in at most as many images as y and by the assumption that x and y are not connected we have $|\text{Im}(V)| \geq |\text{Im}(U)|/2$. But $|U| \leq |\text{Im}(U)| |C_r(\Gamma_U)|$ and so it remains to prove that $|C_r(\Gamma_V)| \geq 2 |C_r(\Gamma_U)|$.

Since Γ_V is a subgraph of Γ_U every r -colouring of Γ_V is an r -colouring of Γ_U . Let $\kappa \in C_r(\Gamma_U)$, let $c \in C \setminus \{x\kappa\}$ be fixed, and define a mapping $\theta_\kappa : \{1, 2, \dots, n\} \rightarrow C$ by

$$m\theta_\kappa = \begin{cases} m\kappa & \text{if } m \neq x, \\ c & \text{if } m = x. \end{cases}$$

If (v, w) is an edge in Γ_V , then since x has degree zero in Γ_V it follows that $v \neq x$ and $w \neq x$. Therefore $v\theta_\kappa = v\kappa \neq w\kappa = w\theta_\kappa$. This implies that θ_κ is a colouring of Γ_V and $\kappa \neq \theta_\kappa$.

Let $\kappa_1, \kappa_2 \in C_r(\Gamma_U)$ such that $\theta_{\kappa_1} = \theta_{\kappa_2}$. Then κ_1 and κ_2 agree on every vertex of Γ_U except possibly x . However, in the graph Γ_U , x is contained in a subgraph that is isomorphic to K_r . Thus the colour of x is uniquely determined by the colours of the other vertices in the complete subgraph and so we conclude that $\kappa_1 = \kappa_2$.

It remains to prove that each of these new colourings θ_κ is distinct from all the colourings in $C_r(\Gamma_U)$. Let $\kappa_1, \kappa_2 \in C_r(\Gamma_U)$. If θ_{κ_1} agrees with κ_2 on $\{1, 2, \dots, n\} \setminus \{x\}$, then $\kappa_1 = \kappa_2$ as in the previous paragraph. Hence $x\theta_{\kappa_1} \neq x\kappa_1 = x\kappa_2$ and so $\theta_{\kappa_1} \neq \kappa_2$.

To conclude, we have shown that $C_r(\Gamma_U) \cup \{\theta_\kappa : \kappa \in C_r(\Gamma_U)\} \subseteq C_r(\Gamma_V)$ and $|C_r(\Gamma_U) \cup \{\theta_\kappa : \kappa \in C_r(\Gamma_U)\}| = 2|C_r(\Gamma_U)|$. \square

Proof of Theorem 4.1. Let U be the largest completely simple subsemigroup of \mathcal{T}_n with $\text{Im}(U) = \{\{1, 2, \dots, r\}\}$. From Proposition 3.1 and the discussion at the beginning of this section, $|U| = r!r^{n-r}$. By repeated application of Lemma 4.4, it follows that the size of a completely simple subsemigroup of \mathcal{T}_n contained in $D(r)$ is no larger than $|U|$. \square

Theorem 4.1 gives a class of examples of completely simple semigroups with largest possible size. The elements of any semigroup in this class all have the same image. However, not every completely simple semigroup with largest size has this form, as the following examples show.

Example 4.5. The mappings

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 3 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 3 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 3 & 4 & 4 \end{pmatrix},$$

generate a completely simple subsemigroup $U \leq \mathcal{T}_5$ with two \mathcal{L} -classes and four \mathcal{R} -classes and size $16 = 2!2^3$.

Example 4.6. The mappings

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 2 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 3 & 2 & 2 \end{pmatrix},$$

generate a completely simple subsemigroup $U \leq \mathcal{T}_5$ with two \mathcal{L} -classes and four \mathcal{R} -classes and size $16 = 2!2^3$.

Example 4.5 differs from Example 4.6 in the fact that the two images in the former have empty intersection whereas the images in the latter have one element in common.

Example 4.7. The mappings

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 2 & 1 & 2 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 3 & 2 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 3 & 4 & 3 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 1 & 4 & 4 & 4 \end{pmatrix},$$

generate a completely simple subsemigroup $U \leq \mathcal{T}_6$ with four \mathcal{L} -classes and four \mathcal{R} -classes and size $32 = 2!2^4$.

5. Inverse semigroups

In this section we find the largest possible size of an inverse subsemigroup of \mathcal{T}_n .

Theorem 5.1. Let U be an inverse subsemigroup of \mathcal{T}_n where the least rank of an element is r . Then

$$|U| \leq r! + \sum_{m=1}^{n-r} \binom{n-r}{m}^2 (r-1)! m!.$$

Theorem 5.3 shows that the upper bound given in Theorem 5.1 is actually attained by an inverse subsemigroup of \mathcal{T}_n .

Before giving the proof of Theorem 5.1 we give a technical lemma, the conclusion of which we require, but whose proof is not particularly enlightening.

Lemma 5.2. Let $m, r \in \mathbb{N}$, $m, r \geq 1$, and let $M(m, r)$ be the maximum value of $\prod_{k_i \neq 0} k_i!(i-1)!$ where the numbers $k_1, k_2, \dots, k_{m+1} \in \mathbb{N}$ partition r and $\sum_{i=1}^{m+1} i k_i = m + r$. Then $M(m, r) \leq m!(r-1)!$.

Proof. We proceed by induction on m . If $m = 1$, then $M(1, r) = (r - 1)!$ (assuming, of course, the convention that $0! = 1$). Assume that $N \geq 2$ and that for any r and any $m < N$ we have $M(m, r) \leq m!(r - 1)!$. Let $l_1, l_2, \dots, l_{N+1} \in \mathbb{N}$ such that $\sum_{i=1}^{N+1} l_i = r$ and $\sum_{i=1}^{N+1} i l_i = N + r$.

If $l_1 + l_i = r$, for some $l_1, l_i > 0$, then $N + r = l_1 + i l_i$ and so $N = (i - 1) l_i$. Hence $l_1! l_i! (i - 1)! \leq l_1! (l_i (i - 1))! = l_1! N! \leq (r - 1)! N!$.

On the other hand, if there exist $i > j > 1$ such that $l_i \neq 0$ and $l_j \neq 0$, then, by the inductive hypothesis,

$$\prod_{\substack{l_k \neq 0 \\ k \neq i}} l_k! (k - 1)! \leq M(N - (i - 1) l_i, r - l_i) \leq (N - (i - 1) l_i)! (r - l_i - 1)!.$$

(Note that the inductive hypothesis applies in this case because $N - (i - 1) l_i \neq 0$ or N). Hence $\prod_{l_k \neq 0} l_k! (k - 1)! \leq l_i! (i - 1)! (N - (i - 1) l_i)! (r - l_i - 1)! \leq (r - 1)! (N - (i - 1) l_i + (i - 1))!$. Since $l_i \geq 1$ and l_1, l_2, \dots, l_{N+1} were arbitrary, it follows that $M(N, r) \leq N! (r - 1)!$. \square

Proof of Theorem 5.1. The least \mathcal{D} -class $U \cap D(r)$ of U is a group and so, by Proposition 2.2, $\text{im}(\alpha) = \text{im}(\beta)$ and $\ker(\alpha) = \ker(\beta)$ for all $\alpha, \beta \in U \cap D(r)$. Moreover, $|U \cap D(r)| \leq r!$. Let $\text{im}(U \cap D(r))$ denote the image, and $\ker(U \cap D(r))$ denote the kernel, of any (and hence every) element in $U \cap D(r)$.

Let $\alpha \in U \cap D(r)$. Then for all $\beta \in U$ we have $\alpha\beta, \beta\alpha \in U \cap D(r)$. Hence if $i \in \text{im}(U \cap D(r))$, then $i\beta \in \text{im}(U \cap D(r))$. In other words, $\beta|_{\text{im}(U \cap D(r))}$ is a permutation of $\text{im}(U \cap D(r))$. Moreover, if $(i, j) \in \ker(U \cap D(r)) = \ker(\beta\alpha)$, then $i\beta\alpha = j\beta\alpha$, and so $(i\beta, j\beta) \in \ker(U \cap D(r))$. In this way, every element $\beta \in U$ is said to *preserve* $\text{im}(U \cap D(r))$ and $\ker(U \cap D(r))$.

Let G be a subgroup of U contained in $D(r + m)$ for some $m \geq 1$ and denote by $\text{im}(G)$ and $\ker(G)$ the image and kernel, respectively, of any element in G . Let K be a class of $\ker(U \cap D(r))$. Then $|K \cap \text{im}(G)| = i \geq 1$ since $\text{im}(U \cap D(r)) \subseteq \text{im}(G)$ and $|K \cap \text{im}(U \cap D(r))| = 1$. If $\gamma \in G$, then since $\gamma|_{\text{im}(G)}$ is a permutation of $\text{im}(G)$ it follows that $(K \cap \text{im}(G))\gamma$ contains i elements. Moreover, since γ preserves $\ker(U \cap D(r))$ we deduce that $(K \cap \text{im}(G))\gamma$ is contained in a single class of $\ker(U \cap D(r))$. Thus for each i , γ permutes the set of all classes K in $\ker(U \cap D(r))$ with $|K \cap \text{im}(G)| = i$. If there are k_i such classes, then there are $k_i!$ such permutations. Let K and L be classes of $\ker(U \cap D(r))$ such that $|K \cap \text{im}(G)| = |L \cap \text{im}(G)| = i$. Then there are $(i - 1)!$ bijections from $K \cap \text{im}(G)$ to $L \cap \text{im}(G)$ that preserve $\text{im}(U \cap D(r))$. It follows, by Lemma 5.2, that

$$|G| \leq \prod_{k_i \neq 0} k_i! (i - 1)! \leq m! (r - 1)!$$

since $\sum_{i=1}^{m+1} k_i = r$ and $\sum_{i=1}^{m+1} i k_i = m + r$.

Finally, there are at most $\binom{n-r}{m}$ groups in $U \cap D(r + m)$, one for each subset of $\{1, 2, \dots, n\}$ of size $r + m$ containing $\text{im}(U \cap D(r))$. This implies that

$$|U \cap D(r + m)| \leq \binom{n-r}{m}^2 m! (r - 1)!$$

and the proof is complete. \square

We now demonstrate that there exist inverse subsemigroups of \mathcal{T}_n with size $r! + \sum_{m=1}^{n-r} \binom{n-r}{m}^2 (r - 1)! m!$ for each r . Let V_r be any generating set for the subgroup of \mathcal{T}_n consisting of all mappings with image $\{1, 2, \dots, r\}$ and kernel $\{\{1\}, \{2\}, \dots, \{r - 1\}, \{r, r + 1, \dots, n\}\}$, let W_r be any generating set for the subgroup of \mathcal{T}_n consisting of all π such that $\{1, \dots, r - 1\}\pi = \{1, \dots, r - 1\}$ and $\{r + 1, \dots, n\}\pi = \{r + 1, \dots, n\}$, and let

$$\alpha_r = \begin{pmatrix} 1 & 2 & \cdots & n - 1 & n \\ 1 & 2 & \cdots & n - 1 & r \end{pmatrix}.$$

Clearly, $\langle V_r \rangle \cong \mathcal{S}_r$ and $\langle W_r \rangle \cong \mathcal{S}_{r-1} \times \mathcal{S}_{n-r}$.

Theorem 5.3. The semigroup $U(n, r) = \langle V_r, W_r, \alpha_r \rangle$ is an inverse subsemigroup of \mathcal{T}_n with size $r! + \sum_{m=1}^{n-r} \binom{n-r}{m}^2 (r - 1)! m!$.

Proof. We start by showing that the only elements of $U(n, r)$ with rank r are those in $\langle V_r \rangle$. Let $\mu \in U(n, r) \setminus \langle V_r \rangle$. Then μ can be given as the product $x_1 x_2 \cdots x_k$ of elements $x_1, x_2, \dots, x_k \in V_r \cup W_r \cup \{\alpha_r\}$. In fact, since $\langle V_r \rangle$ is an ideal in $U(n, r)$ it follows that $x_1, x_2, \dots, x_k \in W_r \cup \{\alpha_r\}$. By a simple inductive argument on k , we see that $\text{rank}(\mu) = n$ or $r\mu^{-1}$ is the unique non-trivial class of $\ker(\mu)$. Thus it follows quickly that $\text{rank}(\mu) > r$.

Since $\langle V_r \rangle$ is a group, it follows that $U(n, r) \cap D(r)$ is regular. Secondly, if $\mu \in U(n, r) \setminus \langle V_r \rangle$, it is possible to find $v \in \langle W_r \rangle$ such that $\mu v \mu = \mu$, and so $U(n, r)$ is regular.

Each of the transpositions $(n \ r \ m+1)$ lies in $\langle W_r \rangle$ for $0 \leq m \leq n-r-2$. Thus $\text{im}(\alpha_r(n \ n-1)\alpha_r(n \ n-2) \cdots \alpha_r(n \ r+m+1)\alpha_r) = \{1, \dots, r, r+1, \dots, r+m\} \in \text{Im}(U(n, r))$. Acting by elements of W_r on this set we deduce that $\text{Im}(U(n, r))$ consists of all subsets of $\{1, 2, \dots, n\}$ containing $\{1, 2, \dots, r\}$. Likewise, $\ker(\alpha_r(n \ n-1)\alpha_r(n \ n-2) \cdots \alpha_r(n \ r+m+1)\alpha_r) = \{\{1\}, \{2\}, \dots, \{r-1\}, \{r+1\}, \dots, \{r+m\}, \{r, r+m+1, \dots, n\}\}$. Again acting by elements of W_r on this kernel we deduce that $\text{Ker}(U(n, r))$ consists of all kernels with at most one class of size greater than 1 which, if it exists, contains r and is contained in $\{r, \dots, n\}$.

Having determined the images and kernels that elements of $U(n, r)$ admit we see that $|\text{Im}(U(n, r))| = |\text{Ker}(U(n, r))|$. Moreover, for each kernel there is only one image that is a transversal of this kernel. Thus, by Proposition 2.6, $U(n, r)$ is an inverse semigroup.

Any subgroup of $U(n, r)$ with image size $m+r$ has $(r-1)!m!$ elements, and so $U(n, r)$ has the required size. \square

Example 5.4. Let

$$V_5 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 4 & 5 & 5 & 5 \end{pmatrix} \right\},$$

$$W_5 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 1 & 5 & 6 & 7 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 7 & 6 \end{pmatrix} \right\},$$

$$\alpha_5 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 5 \end{pmatrix}.$$

Then $\langle V_5, W_5, \alpha_5 \rangle$ is an inverse subsemigroup of \mathcal{T}_7 with 264 elements.

Let $U(n, 1)$ be the semigroup generated by the transformations

$$W_1 = \left\{ \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ 1 & 3 & 2 & 4 & \cdots & n \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 3 & 4 & \cdots & n & 2 \end{pmatrix} \right\},$$

$$\alpha_1 = \begin{pmatrix} 1 & 2 & 3 & \cdots & n-1 & n \\ 1 & 2 & 3 & \cdots & n-1 & 1 \end{pmatrix}.$$

Then $U(n, 1)$ is isomorphic to the symmetric inverse semigroup \mathcal{I}_{n-1} on an $(n-1)$ -element set. Note that $U(n, 1)$ is the semigroup described before Theorem 5.3 and in this case $V_1 = \emptyset$.

Corollary 5.5. *The largest possible size of an inverse subsemigroup of \mathcal{T}_n is $\sum_{m=0}^{n-1} \binom{n-1}{m}^2 m!$ and $U(n, 1)$ is an inverse semigroup with this size.*

Proof. Since $r! + \sum_{m=1}^{n-r} \binom{n-r}{m}^2 (r-1)!m! \leq \sum_{m=0}^{n-1} \binom{n-1}{m}^2 m!$ for all $r \in \{1, \dots, n\}$ the result follows. \square

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